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Coherent states for some quantum superalgebras

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Abstract : The coherent states for the q-deformations of the supersymmetric oscillator and of the superalgebra *Osp* (1,2) are obtained. We start from the boson-fermion realization of the undeformed systems and, replacing the operators by their quantum counterparts, get the q-deformed supersymmetric annihilation operators. The q-coherent states are constructed as eigenstates of one or more of these annihilation operators. They may be expressed as a sum of a part having a pure q-bosonic nature and a fully supersymmetric part (q-deformed)

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The quantum deformations of the coherent states for different Lie algebras have been considered by several authors [1–4]. One very useful procedure is to start with available boson realization of the Lie algebra, replace the boson operators by q-bosonic ones and get the coherent states as the eigenstates of one or more q-annihilation operators. In view of the fact that the interest in quantum algebras has been extended to quantum superalgebras [5–7], we propose to construct here the coherent states for some q-deformed supersymmetric algebras. We begin with the case of the q-deformed supersymmetric oscillator whose Hamiltonian is given by (the q-deformed operators are denoted by the tilde over the undeformed operators)

$$\tilde{H} = \frac{1}{2}\,\omega(\tilde{a}^\dagger\tilde{a} + \tilde{a}\,\tilde{a}^\dagger + \tilde{f}^\dagger\tilde{f} - \tilde{f}\tilde{f}^\dagger) \tag{1}$$

where the q-boson and q-fermion annihilation and creation operators $\tilde{a}$, $\tilde{a}^\dagger$, $\tilde{f}$, $\tilde{f}^\dagger$ satisfy

$$\tilde{a}\,\tilde{a}^\dagger - q^{1/2}\tilde{a}^\dagger\tilde{a} = q^{-N_b/2}, \tag{2}$$

$$\tilde{f}\tilde{f}^\dagger + q^{1/2}\tilde{f}^\dagger\tilde{f} = q^{N_f/2} \tag{3}$$

$$\tilde{f}\tilde{f} = \tilde{f}^\dagger\tilde{f}^\dagger = 0. \tag{4}$$

Here, the number operators N_b and N_f for the q-deformed bosons and fermions are the same [8] as in the undeformed case. It may be remembered that

$$\tilde{a}\tilde{a}^{\dagger} = [N_b + 1], \quad \tilde{a}^{\dagger}\tilde{a} = [N_b], \tag{5}$$

where
$$[x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}}. \tag{6}$$

The eigenvalues of N_b and N_f will be denoted by n_b (0, 1, 2, 3, ...) and n_f (0, 1) respectively. As the parameter $q \to 1$, a quantum algebra (superalgebra) reduces to the corresponding Lie (graded Lie) algebra. Because of the reason that N_f has only two eigenvalues 0 and 1, the quantum deformed fermion operators operating on the relevant states become identical with the undeformed fermion operators. We shall use a representation in which

$$f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f^{\dagger} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{7}$$

so that states with no fermion and one fermion are represented by

$$\begin{pmatrix} |n_b> \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ |n_b> \end{pmatrix}, \tag{8}$$

respectively. Following Aragone and Zypman [9], who investigated the undeformed supersymmetric oscillator, we now define the q-analogue of the supersymmetric annihilation operator as

$$\tilde{A} = \begin{pmatrix} \tilde{a} & 1 \\ 0 & \tilde{a} \end{pmatrix}. \tag{9}$$

It may be noted, however, that Aragone and Zypman used a representation in which the matrices for f and $f^{\dagger}$ shown in eq. (7) are interchanged. As a consequence, a purely bosonic state is represented as $\begin{pmatrix} 0 \\ |n\rangle \end{pmatrix}$ in their paper. In our representation, the annihilation operator annihilates the ground state with $n_b = 0$, $n_f = 0$. The eigenstates of $\tilde{A}$ are the coherent states of the q-deformed supersymmetric oscillator. These coherent states are given by

$$|\phi\rangle_a = \begin{pmatrix} c|\alpha\rangle_q - d\frac{\partial}{\partial\alpha}|\alpha\rangle_q \\ d|\alpha\rangle_q \end{pmatrix}, \tag{10}$$

where c, d are constants, $|\alpha>_q$'s are the unnormalized q-bosonic coherent states [1],

$$|\alpha\rangle_q = \sum_{n=0} \frac{\alpha^n}{([n]!)^{1/2}} |n\rangle, \tag{11}$$

with the q-factorial defined by

$$[n]! = [n][n-1] \,\ldots\ldots\ldots\, [1] \tag{12}$$

and α is any complex number. It can be easily verified that

$$\tilde{A}\,|\phi\rangle_a = \alpha|\phi\rangle_a. \tag{13}$$

Using further a procedure similar to that in reference [9], the quantum supercoherent state of eq. (10) may be written in an alternative form,

$$|\phi\rangle_q = c_1|\phi_b\rangle_q + d_1|\phi_s\rangle_q, \tag{14}$$

where
$$|\phi_b\rangle_q = \begin{pmatrix} |\alpha\rangle_q \\ 0 \end{pmatrix}, \tag{15}$$

and
$$|\phi_s\rangle_q = \begin{pmatrix} \alpha^*\left(e_q\left(|\alpha|^2\right)\right)^{-1}\left(\frac{\partial}{\partial|\alpha|^2}e_q\left(|\alpha|^2\right)\right)|\alpha\rangle_q - \frac{\partial}{\partial\alpha}|\alpha\rangle_q \\ |\alpha\rangle_q \end{pmatrix} \tag{16}$$

Here $e_q\left(|a|^2\right)$ is a q-exponential function defined by

$$e_q(|z|) = \sum_{n=0}^{\infty}\frac{z^n}{[n]!}. \tag{17}$$

One can check that the states $|\phi_b>_q$ and $|\phi_s>_q$ are orthogonal to each other. $|\phi_b>_q$ has pure q-bosonic nature with properties similar to the q-coherent states discussed by Gray and Nelson [2]. Further, as $q \to 1$, these states correspond to minimum uncertainty allowed by Heisenberg uncertainty principle and, in this limit, may be thought of as more classical [9]. In contrast, the state $|\phi_s>_q$ is the q-analogue of a fully supersymmetric coherent state. We want to point out that because of the difference in definitions used by Aragone and Zypman [9] as mentioned earlier, they had, for the undeformed problem, a purely fermionic state to behave more classically. Remembering this difference, the $q \to 1$ limit of the state given by eq. (14) is analogous to that obtained by these authors.

We next proceed to construct the coherent states for the q-deformation of the superalgebra $Osp(1,2)$. The even and odd generators of the undeformed $Osp(1,2)$ may be expressed [10] as combinations of two pairs of bosonic annihilation and creation operators a_i, $a_i^\dagger (i = 1,2)$ and one pair of fermionic annihilation and creation operators $f, f^\dagger$:

$$J_+ = a_1^\dagger a_2,\ J_- = a_2^\dagger a_1,\ J_3 = 1/2(N_1 - N_2),$$
$$R_+ = -1/2\left(a_1^\dagger f + a_2 f^\dagger\right),\ R_- = -1/2\left(a_2^\dagger f - a_1 f^\dagger\right), \tag{18}$$

where N_1 and N_2 are the number operators of the two kinds of bosons. With the standard commutation and anticommutation relations among the three pairs of annihilation and creation operators, it can be readily shown that $J_\pm$, J_3, $R_\pm$ satisfy the commutation and anticommutation relations of the $Osp(1,2)$ superalgebra [10]. The irreducible representations for this algebra are characterized by a quantum number λ which can take integral and half integral values. If n_1, n_2 and n_f are the eigenvalues of the number operators for the two types of bosons and the fermions, then

$$\lambda = 1/2(n_1 + n_2 + n_f). \tag{19}$$

Writing the 'angular momentum' labels

$$j = 1/2(n_1 + n_2),\ m = 1/2(n_1 - n_2), \tag{20}$$

the vectors spanning the representation space are

$$\left|n_1, n_2, n_f\right\rangle \equiv |\lambda; j, m\rangle. \tag{21}$$

We note that $j = 0$ for $\lambda = 0$ and for any value of $\lambda > 0$, j is either equal to λ (for $n_f = 0$) or is $\lambda - 1/2$ (for $n_f = 1$). The action of the operators $J_\pm$, J_3, $R_\pm$ on the states $|\lambda; j = \lambda, m>$ and $|\lambda; j = \lambda - 1/2, m>$ may be explicitly seen through the actions of the operators a_i, $a_i^\dagger (i = 1,2)$, $f, f^\dagger$ on the states $|n_1, n_2, n_f>$. In analogy with the case of the supersymmetric oscillator, a purely bosonic state ($n_f = 0$) will henceforth be represented as

$$\begin{pmatrix} |n_1\rangle\,|n_2\rangle \\ 0 \end{pmatrix}. \tag{22}$$

We now define two supersymmetric annihilation operators

$$A_1 = \begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix},\ A_2 = \begin{pmatrix} a_2 & 1 \\ 0 & a_2 \end{pmatrix}. \tag{23}$$

Before going to the quantum deformation of *Osp* (1,2), let us construct the coherent states for the undeformed *Osp* (1,2) algebra. These are the common eigenstates of A_1 and A_2 and are given by

$$|\psi\rangle = \begin{pmatrix} c|\alpha_1\rangle\,|\alpha_2\rangle - d\dfrac{\partial}{\partial\alpha_1}|\alpha_1\rangle\,|\alpha_2\rangle - d|\alpha_1\rangle\dfrac{\partial}{\partial\alpha_2}|\alpha_2\rangle \\ d|\alpha_1\rangle\,|\alpha_2\rangle \end{pmatrix}, \tag{24}$$

where $|\alpha_1>$ and $|\alpha_2>$ are the two unnormalized bosonic coherent states. Since the supercoherent states $|\psi>$ are constructed by extension of the method for the supersymmetric oscillator [9], their properties are transparent and need not be discussed any more. We shall now q-deform the superalgebra *Osp* (1,2) and get the corresponding coherent states. This is simply done by replacing $a_i, a_i^\dagger (i = 1,2)$ and $f, f^\dagger$ by their quantum deformed counterparts, satisfying

$$\begin{aligned} &[\tilde{a}_i, \tilde{a}_j] = [\tilde{a}_i^\dagger, \tilde{a}_j^\dagger] = 0 \qquad (i = 1, 2;\ j = 1, 2) \\ &[\tilde{a}_i, \tilde{a}_j^\dagger] = 0 \text{ if } i \neq j \\ &\tilde{a}_i \tilde{a}_i^\dagger - q^{1/2}\tilde{a}_i^\dagger \tilde{a}_i = q^{-N_i/2}. \end{aligned} \tag{25}$$

The anticommutation relations satisfied by $\tilde{f}$ and $\tilde{f}^\dagger$ are the same as given by eqs. (3) and (4). The generators for the q-deformed $Osp\,(1,2)$ are obtained from

$$\begin{aligned}
\tilde{J}_+ &= \tilde{a}_1^\dagger \tilde{a}_2,\ \tilde{J}_- = \tilde{a}_2^\dagger \tilde{a}_1,\ J_3 = 1/2(N_1 - N_2), \\
\tilde{R}_+ &= -1/2\left(\tilde{a}_1^\dagger \tilde{f} + \tilde{a}_2 \tilde{f}^\dagger\right),\ \tilde{R}_- = -1/2\left(\tilde{a}_2^\dagger \tilde{f} - \tilde{a}_1 \tilde{f}^\dagger\right),
\end{aligned} \tag{26}$$

with the following commutation and anticommutation relations (it may be noted that J_3 in eq. (26) is the same as J_3 without deformation),

$$\begin{aligned}
&\left[J_3, \tilde{J}_\pm\right] = \pm \tilde{J}_\pm, \left[\tilde{J}_+, \tilde{J}_-\right] = \left[2J_3\right], \\
&\left[J_3, \tilde{R}_\pm\right] = \pm 1/2 \tilde{R}_\pm, \left[\tilde{J}_\pm, \tilde{R}_\pm\right] = 0, \\
&\left[J_\pm, \tilde{R}_\mp\right] = f_q\left(N_1, N_2\right)\tilde{R}_\pm + g_q\left(N_1, N_2\right)\tilde{R}_\mp^\dagger, \\
&\left\{\tilde{R}_\pm, \tilde{R}_\pm\right\} = \pm 1/2 \tilde{J}_\pm, \left\{\tilde{R}_+, \tilde{R}_-\right\} = -1/4\left([N_1] - [N_2]\right) - 1/2 g_q(N_1, N_2) N_f,
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
f_q(N_1, N_2) &= 1/2\left([N_1 + 1] + [N_2 + 1] - [N_1] - [N_2]\right), \\
g_q(N_1, N_2) &= 1/2\left([N_1 + 1] - [N_2 + 1] - [N_1] + [N_2]\right).
\end{aligned} \tag{28}$$

The operators $\tilde{J}_\pm$, J_3, $\tilde{R}_\pm$, operate on states given by eq. (21). It may be mentioned here that some variants of the generating elements introduced in eqs. (26) are also in use in the literature [5–7,10] for defining a quantum deformation of the superalgebra $Osp\,(1,2)$ often called $Osp_q\,(1,2)$ or $U_q\,[Osp\,(1,2)]$. While some of the corresponding commutation and anticommutation relations are the same as given in eq. (27), others are different to some extent. In the limit $q \to 1$ and recalling eq. (6), the relations (27) reduce to the corresponding relations for undeformed $Osp\,(1,2)$.

In order to construct the coherent states for the q-deformation of the superalgebra $Osp\,(1,2)$, we follow Biedenharn's idea [1] of defining q-coherent states as the eigenstates of the q-annihilation operators and generalize it to the supersymmetric case. We then start with the pair of q-modified supersymmetric annihilation operators

$$\tilde{A}_i = \begin{pmatrix} \tilde{a}_i & 1 \\ 0 & \tilde{a}_i \end{pmatrix} \qquad (i = 1,2) \tag{29}$$

and find the common eigenstates $|\psi>_q$ of $\tilde{A}_1$ and $\tilde{A}_2$. Writing the unnormalized common eigenkets of $\tilde{a}_1$ and $\tilde{a}_2$ belonging to the complex eigenvalues α_1 and α_2, respectively, as [4]

$$\begin{aligned}
\left|\alpha_1, \alpha_2\right\rangle_q &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{\alpha_1^{n_1} \alpha_2^{n_2}}{\left([n_1]!\,[n_2]!\right)^{1/2}} \left|n_1, n_2\right\rangle \\
&= \sum_j \sum_{m=-j}^{j} \frac{\alpha_1^{j+m} \alpha_2^{j-m}}{\left([j+m]!\,[j-m]!\right)^{1/2}} \left|j, m\right\rangle,
\end{aligned} \tag{30}$$

$|\psi>_q$ is given by an expression which is an extension of the q-coherent state of the supersymmetric oscillator,

$$|\psi\rangle_q = c_1|\dot{\psi}_b\rangle_q + d_1|\psi_s\rangle_q, \tag{31}$$

with a pure q-bosonic part $|\psi_b>_q$,

$$|\psi_b\rangle_q = \begin{pmatrix} |\alpha_1, \alpha_2\rangle_q \\ 0 \end{pmatrix}, \tag{32}$$

and another part which is the quantum analogue of the fully supersymmetric coherent state,

$$\begin{aligned} |\psi_s\rangle_q = \\ \{\alpha_1^*[e_q(|\alpha_1|^2)]^{-1}[\frac{\partial}{\partial|\alpha_1|^2} e_q(|\alpha_1|^2)] + \alpha_2^*[e_q(|\alpha_2|^2)]^{-1}[\frac{\partial}{\partial|\alpha_2|^2} e_q(|\alpha_2|^2)]\}\bigg| \\ \times |\alpha_1, \alpha_2>_q - \frac{\partial}{\partial\alpha_1}|\alpha_1, \alpha_2>_q - \frac{\partial}{\partial\alpha_2}|\alpha_1, \alpha_2>_q \\ |\alpha_1, \alpha_2>_q \end{aligned} \tag{33}$$

Again the states $|\psi_b>_q$ and $|\psi_s>_q$ are orthogonal to each other. The states $|\psi_b>_q$ are related to the q-analogue of angular momentum coherent states [4] which as $q \to 1$, reduces to the angular momentum coherent states of Atkins and Dobson [11]. The expectation values of the angular momentum operators for these states may be calculated easily. As in the case of the quantum deformed supersymmetric oscillator, the state $|\psi_s>_q$ is a fully supersymmetric q-coherent state.

In conclusion, we like to mention that we have seen in our earlier work [4] the advantage of using the boson realization of some Lie algebras to get the quantum deformation and construct the corresponding coherent states. Similar advantage is there in the use of the boson-fermion realization of the superalgebras. The quantum deformation is straightway obtained by q-deforming the boson and fermion operators. The quantum analogues of the supersymmetric coherent states for different superalgebras are constructed as eigenvectors of one or more q-deformed supersymmetric annihilation operators. The coherent states, generally, may be written as a sum of a part having a pure q-bosonic nature and a fully supersymmetric part, the two parts being mutually orthogonal. As $q \to 1$, the first part represents a state with more classical behaviour.

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